Even Subdivison-Factors of Cubic Graphs

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Abstract

We call a set $\mathcal{S}$ of graphs an "even subdivison-factor" of a cubic graph G if G contains a spanning subgraph H such that every component of H has an even number of vertices and is a subdivision of an element of $\mathcal{S}$. We show that any set of 2-connected graphs which is an even subdivison-factor of every 3-connected cubic graph, satisfies certain properties. As a consequence, we disprove a conjecture which was stated in an attempt to solve the circuit double cover conjecture.

Keywords: circuit double cover, factor, frame, Petersen graph

1 Basic definitions and main results

For terminology not defined here we refere to [1]. There are several ways to describe that a spanning subgraph with certain properties exists in a cubic graph G.
A set S of graphs is called a *component-factor* of G if G has a spanning subgraph H such that every component of H is an element of S, see [6]. Within the topic of circuit double covers the notion of a *frame* was introduced, see [3, 4, 7, 8]. Some slightly different definitions of a frame exist. Here, a frame of G is a graph F where every component of F is either an even circuit or a 2-connected cubic graph such that the following holds: G has a spanning subgraph F' which is a subdivision of F and every component of F' has an even number of vertices. For our purpose it is useful to join these two concepts.

Definition 1.1 *A set $\mathcal{S}$ of graphs is called a subdivison-factor of a cubic graph G if G contains a spanning subgraph H such that every component of H is a subdivision of an element of $\mathcal{S}$. If every component of H has an even number of vertices then $\mathcal{S}$ is called an even subdivision-factor of G.*

*supported by the FWF project P20543.

Example 1.2 *Every* 3*-edge colorable cubic graph* G_3 *has a spanning subgraph consisting of even circuits, i.e. an even* 2*-factor. Hence,* $\{C_2\}$ *where* C_2 *denotes the circuit of length* 2*, is an even subdivision-factor of* G_3*. Reversely, if* $\{C_2\}$ *is an even subdivision-factor of a cubic graph* G*, it follows that* G *is* 3*-edge colorable.*

Thus an even subdivision-factor is a generalization of an even 2-factor. It was asked in a preprint of [4] whether $\{C_2\} \cup \mathcal{H}$ where $\mathcal{H}$ is a certain infinite family of hamiltonian cubic graphs, is an even subdivision-factor of every 3-connected cubic graph. In particular the following is conjectured in [4]. (A cubic graph G which admits a 3-edge coloring such that each pair of color classes forms an hamiltonian circuit, is called a *Kotzig graph*, see [4, 7].)

Conjecture 1.3 *Every 3-connected cubic graph has a spanning subgraph which is a subdivision of a Kotzig graph.*

A positive answer to this conjecture would have solved the circuit double cover conjecture (CDCC), see [4]. For stating the main theorem which provides a negative answer to Conjecture 1.3 and the posed question above, we use two definitions.

Definition 1.4 *Let* H_i*,* $i \in \{1, 2\}$ *be a subgraph of a graph* G *or a subset of* $V(G)$*. Denote by* $[H_1, H_2]$ *the set of all paths with connect a vertex of* H_1 *with a vertex of* H_2*. Then,* $d_G(H_1, H_2)$ *or in short* $d(H_1, H_2) := \min_{\alpha \in [H_1, H_2]} |E(\alpha)|$*.*

The parameter $l(G)$ below measures to which extend G is not hamiltonian.

Definition 1.5 *Let* G *be a* 2*-connected graph. Denote by* $U(G)$ *the set of all circuits of* G*. Define*

$$l(G) := \min_{C \in U(G)} \max_{v \in V(G)} d(C, v) \, .$$

Let $\mathcal{S}$ *be a set of 2-connected graphs. Define* $l_m(\mathcal{S}) := \max_{G \in \mathcal{S}} l(G)$ *if this maximum exists; otherwise set* $l_m(\mathcal{S}) := \infty$*.*

Note that in the case of G being hamiltonian, $l(G) = 0$. We state the main result.

Theorem 1.6 *Let* $\mathcal{S}$ *be a set of* 2*-connected graphs which is an even subdivision-factor of every* 3*-connected cubic graph, then* $l_m(\mathcal{S}) = \infty$*.*

Theorem 1.6 implies that there is no finite set of graphs which is an even subdivision factor of every 3-connected cubic graph. Note that Conjecture 1.3 remains open for cyclically 4-edge connected cubic graphs. A positive answer to this version would still solve the CDCC since a minimal counterexample to the CDCC is at least cyclically 4-edge connected. In order to prove Theorem 1.6, we prove Theorem 2.14 which concerns the iterated Petersen graph. From now on, we make preparations for the proof of Theorem 2.14.

2 The iterated Petersen graph

We denote by P_{10} the Petersen graph and we set $P := P_{10} - z$, $z \in V(P_{10})$. The *iterated Petersen graph* which is defined next has already been introduced in [2].

Definition 2.1 *Let G be a graph with $d(v) \in \{2,3\}$, $\forall v \in V(G)$. A P-inflation at $v_0 \in V(G)$ is defined as the following operation: add P to $G - v_0$ and connect each former neighbor of v_0 to one distinct 2-valent vertex of P. $G^0, G^1, G^2, ..., G^k$ with $k \in \mathbb{N}$ and $G^0 := G$, is the sequence of graphs where G^i, $i \in \{1, 2, ..., k\}$ results from G^{i-1} by applying the P-inflation at every vertex in G^{i-1}. We call P_{10}^k for $k \geq 1$ an iterated Petersen graph.*

Obviously, G^k is cubic if G is cubic. If G is not cubic, then G and G^k have the same number of vertices of degree 2. See Figure 1 for an illustration of Def. 2.1. Note that if we remove in the illustration of G^i the dangling edges, we obtain P^{i-1}, $i = 1, 2$.

Definition 2.2 *Let W_k, $k \in \mathbb{N}$ denote the set of the three 2-valent vertices of P^k and set $d_k := \max\{\, d(W_k, v) \mid v \in V(P^k) \,\}$. If a graph X, say, is isomorphic to P^k, then $W_k(X)$ denotes the set of the three 2-valent vertices.*

Proposition 2.3 *Let $k \in \mathbb{N}$, then $d_k = 2^{2k+1} - 1$.*

Proof: The statement obviously holds for $k = 0$. Consider P^k for $k > 0$ and set $j_k := \min\{\, |V(\alpha)| \mid \alpha \in [w_1, w_2] \,\}$ with $\{w_1, w_2\} \subseteq W_k$ and $w_1 \neq w_2$. Let $k \geq 1$, then P^k contains 9 disjoint copies of P^{k-1}. P results from P^k by contracting each of them to a distinct vertex. Hence, every copy P' of P^{k-1} in P^k corresponds to a vertex in P. We say a path α *traverses* $P' \subseteq P^k$ if α contains a subpath $\alpha' \subseteq P'$ which connects two distinct vertices of $W_{k-1}(P')$. Every shortest path in P^k which connects w_1 with w_2, traverses exactly 4

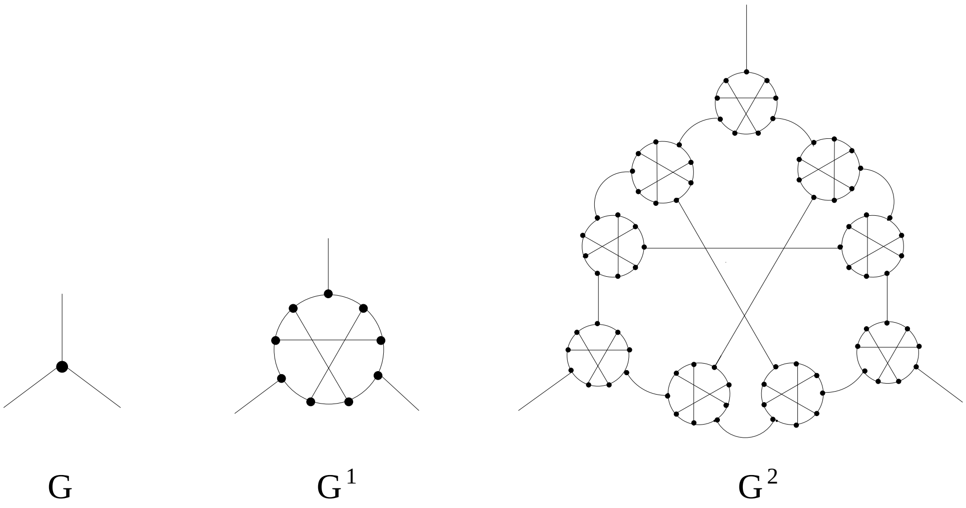

Figure 1: A vertex in a cubic graph G and the corresponding copies of P^{i-1} in G^i, $i = 1, 2$.

copies of P^{k-1} and thus $j_k = 4\, j_{k-1}$. Since $j_0 = 4$, we obtain

$$j_k = 4^{k+1}\,. \qquad (1)$$

Let $k \in \mathbb{N}$. Set $b_k := \max_{v \in V(P^k)} d(w_1, v)$ and $B_k := \{v \in V(P^k) \mid d(v, w_1) = b_k\}$. We claim that

$$B_k = W_k - \{w_1\}\,. \qquad (2)$$

We proceed by induction on k. For $k = 0$, the statement holds. Let $P' \subseteq P^k$ be a copy of P^{k-1} with $v_0 \in B_k \cap V(P')$. Then obviously P' corresponds to a 2-valent vertex of P. Let q_1, q_2 denote the two distinct vertices of P' which form together a vertex cut of P^k and which are both contained in $W_{k-1}(P')$. Then, $d(w_1, q_1) = d(w_1, q_2)$. The induction assumption for $k-1$ on P' implies that $v_0 \in W_{k-1}(P')$. Since $v_0 \notin \{q_1, q_2\}$, $v_0 \in W_k - \{w_1\}$. Hence the claim is proven.

Let $k \geq 1$ and let now $P' \subseteq P^k$ be a copy of P^{k-1} with $x \in V(P')$ and $d(x, W_k) = d_k$, see Def. 2.2. Obviously, P' corresponds to a vertex of degree 3 in P. Let $\alpha_x \subseteq P^k$ connect x with a vertex of W_k and satisfy $|E(\alpha_x)| = d_k$. Hence α_x is a shortest path and traverses exactly one copy of P^{k-1} which corresponds to a 2-valent vertex of P. By applying (2) on P' we conclude that $x \in W_{k-1}(P')$. Thus, $|E(\alpha_x)| = 2j_{k-1} - 1$ which finishes the proof.

Corollary 2.4 *$l(P_{10}) = 1$ and $l(P_{10}^k) = 2^{2k-1}$, $\forall\, k \geq 1$.*

Proof: Since P_{10} has no hamiltonian circuit but $P_{10} - v_0$ is hamiltonian for every $v_0 \in V(P_{10})$, $l(P_{10}) = 1$. Let $k \geq 1$, then P_{10}^k contains ten disjoint copies of P^{k-1} which we denote by X_i, $i = 1, 2, ..., 10$. Ever circuit in P_{10}^k is vertex-disjoint with at least one X_i since otherwise it would imply that P_{10} is hamiltonian. Hence, $l(P_{10}^k) \geq d_{k-1} + 1$. It is not difficult to see that P_{10}^k contains a circuit C which passes through X_i for $i = 1, 2, ..., 9$ and satisfies $W_{k-1}(X_i) \subseteq V(C)$. By the properties of C and since $\bigcup_{i=1}^{10} V(X_i) = V(P_{10}^k)$, it follows that $d(C, v) \leq d_{k-1} + 1$, $\forall v \in V(P_{10}^k)$. Hence, $l(P_{10}^k) = d_{k-1} + 1$ and by applying Prop. 2.3, the proof is finished.

2.1 f-matchings and P-inflations

Definition 2.5 *A matching M of a cubic graph G is called an f-matching if every component of $G - M$ is 2-connected and has an even number of vertices.*

Lemma 2.6 *Suppose a cubic graph G has a minimal 3-edge cut E_0. Then for every f-matching M of G, $|M \cap E_0| \in \{0, 1\}$.*

Proof: Suppose $|M \cap E_0| = 3$. Since E_0 is a minimal edge-cut, $G - E_0$ consists of two components which have both an odd number of vertices. Let L be one of them. Then $L - M$ and thus $G - M$ contains at least one component which has an odd number of vertices, in contradiction to Def. 2.5.
Suppose $|M \cap E_0| = 2$. Then the one edge of E_0 which is not contained in M is a bridge in $G - M$ which contradicts Def. 2.5. Hence the proof is finished.

Lemma 2.7 *Let $E_0 := \{e_1, e_2, e_3\}$ be a minimal 3-edge cut in a 2-connected cubic graph G such that P is one component of $G - E_0$. Then for every f-matching M of G the following is true.*

(1) Consider $P \subseteq G$ as a graph and M restricted to P. Then $P - M$ is connected.

(2) $G - M$ contains a 3-valent vertex within $V(P)$, i.e. at least one vertex of $P \subseteq G$ is not matched by M.

Proof: Let $W_0 := \{w_1, w_2, w_3\}$ denote the set of the 2-valent vertices of P and let $e_i \in E_0$ be incident with w_i, $i = 1, 2, 3$. By Lemma 2.6, $|M \cap E_0| \in \{0, 1\}$.

Proof of the first statement:

Case 1. $|M \cap E_0| = 0$.
All w_i's are contained in the same component L, say, of $P - M$ since otherwise

one component of $G-M$ would have e_i, for some $i \in \{1,2,3\}$ as a bridge in contradiction to Def. 2.5. Suppose by contradiction that $P-M$ has another component L'. Since $V(L') \cap W_0 = \emptyset$, L' is not only a component of $P-M$ but also of $G-M$. By Def. 2.5, $L' \subseteq P$ is 2-connected and thus contains a circuit. There is exactly one circuit C' in P which contains no vertex of W_0, see Figure 1. Then e_i, $i=1,2,3$ is a bridge in $G-M$ contradicting Def. 2.5. Hence $P-M$ is connected.

Case 2. $|M \cap E_0| = 1$. Let w.l.o.g. $M \cap E_0 = \{e_3\}$.
Then w_1 and w_2 are contained in the same component L, say, of $P-M$ otherwise e_i, $i \in \{1,2\}$ is a bridge of $G-M$. Suppose by contradiction that $P-M$ has another component L'. Since e_3 is matched and $w_i \in V(L)$, $i=1,2$, L' is not only a component of $P-M$ but also of $G-M$. By Def. 2.5, L' is 2-connected and thus contains a circuit C'. Since L is a component, L contains a path β (which is vertex-disjoint with C') connecting w_1 with w_2. P_{10} is obtained from P and E_0 by identifying the three endvertices of e_i, $i=1,2,3$ which are not in P. Then β and C' correspond to two disjoint circuits in P_{10} which form a 2-factor of P_{10}. Hence $C' = L'$, and L' is a circuit of length 5 which contradicts Def. 2.5.

Proof of the second statement:

Suppose by contradiction that every vertex of P is matched by M. Since $|V(P)|$ is odd and by Lemma 2.6, $|E_0 \cap M| = 1$. Such matching M covering $V(P)$ corresponds to a perfect matching of P_{10}. Hence, $P-M$ consists of a path and a circuit C of length 5. Then C is also a component of $G-M$ which contradicts Def. 2.5.

Lemma 2.8 *Let G, E_0 and P be as in the previous lemma. Let α be a path in G which passes through P, i.e. α has no endvertex in P and $|E(\alpha) \cap E_0| = 2$. Then for every f-matching M with $E(\alpha) \cap M = \emptyset$ the following is true: $G-M$ contains a 3-valent vertex within $V(\alpha) \cap V(P)$, i.e. at least one vertex of $V(\alpha) \cap V(P)$ is not matched by M.*

Proof: Suppose by contradiction that every vertex of $V(\alpha) \cap V(P)$ is matched by M. Then $\alpha \cap P$ is a component of $P-M$ and thus by Lemma 2.7 (1) the only component of $P-M$. Since $\alpha \cap P$ contains no 3-valent vertex we obtain a contradiction to Lemma 2.7 (2) which finishes the proof.

Proposition 2.9 *Let G be a 2-connected cubic graph and $v_0 \in V(G)$. Denote by G' the cubic graph which is obtained from G by applying the P-inflation at v_0. Then $G'-M'$ is 2-connected for every f-matching M' of G' if and only if $G-M$ is 2-connected for every f-matching M of G.*

Proof: Denote by P' the subgraph of G' which is isomorphic to P and corresponds to $v_0 \in V(G)$.
Suppose by contradiction that M' is an f-matching of G' such that $G' - M'$ is not 2-connected whereas $G - M$ is 2-connected for every f-matching M of G. Set $M_1' := \{e \in M' \,|\, e \notin E(P')\}$. Denote by M_1 the subset of $E(G)$ which corresponds to M_1'. Then,

$$(G' - M')/V(P') = G - M_1 \qquad (3)$$

We show that M_1 is an f-matching. Lemma 2.6 implies that $v_0 \in V(G)$ is covered by at most one edge of M_1. Hence, M_1 is a matching of G. Since $P' - M'$ is connected by Lemma 2.7 (1), equation (3) implies that $G - M_1$ has the same number of components as $G' - M'$. Contracting an edge or shrinking a subset of vertices in a bridgeless graph does not create a bridge. Therefore and since $G' - M'$ is bridgeless by Def. 2.5, equation (3) implies that $G - M_1$ is bridgeless. Every component of $G - M_1$ has a corresponding isomorphic component in $G' - M'$ (and thus an even number of vertices) with the one exception of the component L_0, say, which contains v_0. $P' - M'$ is connected by Lemma 2.7 (1). Denote by L_0' the component of $G' - M'$ with $(P' - M') \subseteq L_0'$. $V(L_0')$ differs from $V(L_0)$ by containing the vertices of $V(P' - M')$ instead of v_0. Since $|V(L_0')|$ is even by Def. 2.5 and both $|V(P' - M')|$ and $|\{v_0\}|$ are odd, $|V(L_0)|$ is even. Hence M_1 is an f-matching of G. Since $G - M_1$ is not 2-connected we obtain a contradiction to the assumption in the beginning.
Suppose by contradiction that M is an f-matching of G such that $G - M$ is not 2-connected whereas $G' - M'$ is 2-connected for every f-matching M' of G'. Denote by M_2' the matching of G' which corresponds to M of G with $E(P') \cap M_2' = \emptyset$. Then M_2' is an f-matching of G'. Since $G' - M_2'$ is not 2-connected we obtain a contradiction which finishes the proof.

Corollary 2.10 *For every f-matching M of P_{10}^k, $k \in \mathbb{N}$, $P_{10}^k - M$ is homeomorphic to a 2-connected cubic graph.*

Proof: $P_{10} - M$ is not a circuit since it would imply that P_{10} is hamiltonian. Therefore and since every bridgeless disconnected subgraph of P_{10} consists of two circuits of length 5, $P_{10} - M$ is homeomorphic to a 2-connected cubic graph. Since P_{10}^k is not hamiltonian and results from P_{10} by P-inflations and since Proposition 2.9 can be applied after each P-inflation, the corollary follows.

2.2 Frames

Lemma 2.11 *Let $k \in \mathbb{N}$, then P_{10}^k is a frame of P_{10}^{k+1}.*

Proof: Let M be a matching of P_{10}^{k+1} such that every copy of P in P_{10}^{k+1} is matched as in Figure 2; M is illustrated by dashed lines. Then M is an f-matching of P_{10}^{k+1} and the cubic graph homeomorphic to $P_{10}^{k+1} - M$ is P_{10}^k. Hence P_{10}^k is a frame of P_{10}^{k+1}.

Definition 2.12 *Let α be a path in a graph G, then $p(\alpha)$ denotes the number of distinct copies of P with which α has a non-empty vertex-intersection. For $H_i \subseteq G$, $i = 1, 2$, we define $p[H_1, H_2] := \min\{p(\alpha) \,|\, \alpha \in [H_1, H_2]\}$ and we set $p_k := \max\{p[v, W_k] \,|\, v \in V(P^k)\}$, $k \in \mathbb{N}$.*

Lemma 2.13 *Let $k \in \mathbb{N}$, then $p_{k+1} = 2^{2k+1}$ and $p_0 = 1$.*

Proof: Clearly, $p_0 = 1$. Let $P(x)$ and $P(y)$ denote two distinct copies of P in P_{10}^{k+1}, $k \in \mathbb{N}$ with $x \in V(P(x))$ and $y \in V(P(y))$. Let x' (y') be the vertex in P_{10}^k which corresponds to $P(x)$ ($P(y)$) by regarding P_{10}^k as the graph which is obtained from P_{10}^{k+1} by contracting every copy of P. Then for every path $\alpha \in [x, y]$ and its corresponding path $\alpha' \in [x', y']$, $p(\alpha) = |V(\alpha')|$. Hence, $p[x, y] \geq d(x', y') + 1$. Since for every given path $\beta' \in [x', y']$, there is a path $\beta \in [x, y]$ with $p(\beta) = |V(\beta')|$, $p[x, y] = d(x', y') + 1$. Therefore, $p_{k+1} = d_k + 1$ (Def. 2.2) and by applying Prop. 2.3 the proof is finished.

Theorem 2.14 *Let $\mathcal{F}(k)$ be the set of frames of P_{10}^k, $k \in \mathbb{N}$, then*

(1) every frame G of P_{10}^k is cubic and 2-connected, and

(2) $\displaystyle\min_{G \in \mathcal{F}(k)} l(G) = \begin{cases} k & \text{for } k \in \{0, 1\} \\ 2^{2k-3} & \text{for } k \geq 2 \end{cases}.$

Proof: Corollary 2.10 implies that every element of $\mathcal{F}(k)$ is cubic and 2-connected. For $k = 0$, the equality above holds since $K_{3,3}$ is a frame of P_{10} and $l(K_{3,3}) = 0$.

Set $Q := P_{10}^k$ with $k \geq 1$. Let M be an f-matching of Q. Denote the 2-connected cubic graph which is homeomorphic to $Q - M$ by $\overline{Q}(k)$. Suppose that M is chosen in such a way that $l(\overline{Q}(k))$ is minimal.

A subgraph of $\overline{Q}(k)$ is denoted by $\overline{H}$, say, and the corresponding subgraph in $Q - M$ and Q by H.

Let $\overline{C}$ be a circuit of $\overline{Q}(k)$ such that $\max\limits_{v \in \overline{Q}(k)} d_{\overline{Q}(k)}(\overline{C}, v) = l(\overline{Q}(k))$. Q contains ten disjoint induced subgraphs isomorphic to P^{k-1}. If we contract each of them to a distinct vertex, we obtain P_{10}. Hence C does not pass through each of them since otherwise it would imply that P_{10} is hamiltonian. Let us denote one copy of P^{k-1} in Q which is vertex-disjoint with C, by X.

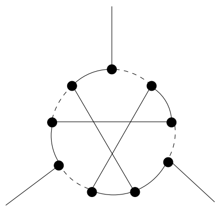

Figure 2: A matching of a copy of P in P^{k+1}.

Let $\{v_1, v_2\} \subseteq V(X)$, then Def. 2.12 implies, if v_1 and v_2 are contained in the same copy of P, that $p[v_1, W_{k-1}(X)] = p[v_2, W_{k-1}(X)]$. Therefore and by Lemma 2.7 (2) there is a vertex $x \in V(X)$ which is not matched by M and which satisfies, $p[x, W_{k-1}(X)] = p_{k-1}$, see Def. 2.12. Denote also by x the corresponding vertex in $\overline{Q}(k)$.

Let $\overline{\alpha}_x \subseteq \overline{Q}(k)$ be a path of length $d(x, \overline{C})$ which connects x with $\overline{C}$.

By the definition of x, $p(\alpha_x) \geq p_{k-1}$. Since $V(C) \cap V(X) = \emptyset$, α_x passes through at least $p_{k-1} - 1$ distinct copies of P. For every such copy of P, $\overline{\alpha}_x$ contains by Lemma 2.8 at least one vertex. Since $\overline{\alpha}_x$ starts and ends in a vertex of degree 3 which is not contained in any of these copies of P, $|V(\overline{\alpha}_x)| \geq p_{k-1} + 1$. Thus and by definition of $\overline{C}$ and $\overline{\alpha}_x$,

$$l(\overline{Q}(k)) \geq d(x, \overline{C}) \geq p_{k-1} \qquad (4)$$

Consider $k = 1$. By inequality (4), $l(\overline{Q}(1)) \geq p_0$. Since $p_0 = 1$ (Lemma 2.13) and since P_{10} is a frame of Q (Lemma 2.11) with $l(P_{10}) = 1$ (Corollary 2.4), $l(\overline{Q}(1)) = 1$.
Consider $k > 1$. By inequality (4) and by Lemma 2.13, $l(\overline{Q}(k)) \geq 2^{2k-3}$. Since by Lemma 2.11, P_{10}^{k-1} is a frame of Q and since by Corollary 2.4 $l(P_{10}^{k-1}) = 2^{2k-3}$, $l(\overline{Q}(k)) = 2^{2k-3}$ which finishes the proof.

Corollary 2.15 *Every P_{10}^k, $k \geq 1$ is a counterexample to Conjecture 1.3.*

Corollary 2.16 *For every set $\mathcal{S}_0$ of 2-connected graphs with $l_m(\mathcal{S}_0) \neq \infty$, there is an infinite set $\mathcal{S}$ of 3-connected cubic graphs with the following property: for every $G \in \mathcal{S}$, $\mathcal{S}_0$ is not an even subdivision-factor of G.*

Proof: Replace every element in $\mathcal{S}_0$ which contains a 2-valent and a 3-valent vertex by its homeomorphic cubic graph. Denote this set by $\mathcal{T}_0$. We observe that if $\mathcal{S}_0$ is an even subdivision-factor of a cubic graph H, say, then $\mathcal{T}_0$ is also an even subdivision-factor of H. Moreover, $l_m(\mathcal{T}_0) \leq l_m(\mathcal{S}_0)$. Set $\mathcal{S} := \{P_{10}^k \mid 2^{2k-3} > l_m(\mathcal{T}_0),\, k \geq 2\}$. Theorem 2.14 implies that for every $G \in \mathcal{S}$, $\mathcal{T}_0$ is not an even subdivision-factor of G. By the above observation, the same holds for $\mathcal{S}_0$ which finishes the proof.

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